ELSEVIER

Journal of Chromatography A, 688 (1994) 1–23

JOURNAL OF CHROMATOGRAPHY A

Theoretical analysis of band profiles in non-linear ideal countercurrent chromatography

GuoMing Zhong[a,b], Georges Guiochon[a,b,*]

[a] *Department of Chemistry, University of Tennessee, Knoxville, TN 37996-1600, USA*

[b] *Division of Analytical Chemistry, Oak Ridge National Laboratory, Oak Ridge, TN 37831-6120, USA*

First received 17 May 1994; revised manuscript received 10 August 1994

Abstract

The analytical solution of the ideal model of chromatography for the band profile of a single component in overloaded elution is extended to the case of the elution of a rectangular pulse in countercurrent moving-bed chromatography. The formation of concentration shocks and diffuse boundaries resulting from the rectangular injection profile of the feed is discussed for compounds moving in the direction of either the liquid phase or the solid phase. Differences between the solutions obtained for the moving-bed and the fixed-bed problems are examined and explained.

1. Introduction

Conventional implementations of chromatography, whether for analytical or preparative applications are batch processes. A feed solution is injected into a continuous stream of mobile phase which percolates through an immobile bed of a suitable stationary phase packed inside a fixed chromatographic column. The feed components are eventually separated and collected or detected at the end of the column. Then, another cycle starts with the injection of a new batch of feed. This same type of cyclic operation takes place in all classic modes of chromatography, elution, displacement or frontal analysis. These processes have been developed to their current state of sophistication by analytical chemists.

In its present form, the elution process satisfies excellently the requirements for chemical analyses and for the production of small amounts of purified products in the research laboratory. It is relatively easy to prepare columns for liquid chromatography that have a high separation power and a throughput commensurate with the range of production rates needed by the pharmaceutical industry. Therefore, the possibility of scaling up a chromatographic process for large-scale industrial production has attracted considerable attention. There are no reasons, however, for the same implementation which suits so well the needs of analytical chemists to be the most convenient for industrial production. Continuous processes have well known advantages over batch processes and much interest has been devoted to the development of a continuous

* Corresponding author. Address for correspondence: Department of Chemistry, University of Tennessee, Knoxville, TN 37996-1600, USA.

0021-9673/94/$07.00
SSDI 0021-9673(94)00765-9

implementation of the chromatographic process [1]. This objective can be achieved if a relative motion is created between the bed of stationary phase and the point at which the feed is introduced into the column. The main solutions proposed so far are countercurrent chromatography (using a sliding column or a "stationary phase" bed moving along the column) and cross-current chromatography, using a rotary annular column [1]. The former approach is generating increasing interest among separation engineers [1–7]. Note that in these implementations it becomes impossible to distinguish the two phases of a chromatographic system as the mobile and the stationary phase, since both phases move along the column. We shall call the former the liquid phase and the latter the solid phase.

The countercurrent effect can be achieved by sliding the column past the feed injection nozzle (sliding column), by moving continuously the solid adsorbent (moving bed, MB) or by using a series of short columns, instead of a long column, and moving sequentially in the direction of the liquid phase the inlet and outlet ports associated with each connection between successive columns (simulated moving bed, SMB) [1–6]. From the perspectives of modeling and initial process design, both true and simulated countercurrent processes can be considered in the same general way [2], and only a general model of countercurrent moving bed needs to be considered at this stage. Numerous theoretical and experimental investigations of MB and SMB have been published, particularly in the recent past [1–6]. However, owing to the complexity of the problem, our understanding of the phenomenon and our ability of optimize the experimental conditions under which the process is run are still limited.

As has happened in fixed-bed chromatography [7], the understanding of the development, migration and progressive separation of the individual bands of the components of a mixture in moving-bed chromatography would be greatly simplified if we knew the evolution of the profiles of single-component bands during their migration in a countercurrent chromatographic system. The use of the ideal model further simplifies considerably the solution of this latter problem by focusing attention on the influence of the thermodynamics of phase equilibrium [7]. As all implementations of chromatography for preparative purposes are bound to operate at high concentrations, only a moderate amount of dispersion distinguishes the band profiles obtained with actual columns from those predicted by the ideal model [7]. The ideal model has been studied in great detail and its analytical solutions for single- and multi-component problems are now well known for elution, frontal analysis and displacement [7–11]. They have been used as a basis not only for a theoretical understanding of the behavior of chromatographic bands, but also for the development of practical procedures of optimization of the experimental conditions for maximum production rate in preparative chromatography [7,12–16]. As we shall show, the ideal model offers similar interest in moving-bed and simulated moving-bed chromatography.

The purpose of this paper is the study of the simplest problem of countercurrent chromatography, the determination of the concentration profile and the migration of overloaded bands of a single component moving in an ideal column (i.e., having an infinite efficiency) under the conditions of moving bed and a pulse feed (with zero feed flow as in the fixed-bed case). This should provide a link between fixed-bed and moving-bed chromatography, a first step toward a detailed understanding of the properties of countercurrent chromatography and the mechanism of multi-component separations under steady-state conditions. In further publications, a binary mixture and a finite feed flow which differentiates continuous from discontinuous operations will be considered.

2. Theory

As explained in the Introduction, countercurrent moving-bed chromatography is very similar to fixed-bed elution, except for the movement of the solid phase and a finite feed flow. However, in this first stage, the effect of the latter is not considered here. Accordingly, the basic assump-

tions of the ideal model for countercurrent chromatography are [7–11] that (i) the solid and the liquid phases are in constant and instantaneous equilibrium; (ii) there is no axial dispersion in the column; (iii) the process is isothermal; (iv) the column is radially homogeneous, and can be considered as one-dimensional; (v) the liquid phase is not compressible; and (vi) the partial molar volume of the component is the same in the liquid and the solid phases and, accordingly, there is no sorption effect [7]. Hence the column is assumed to have an infinite efficiency. Although actual columns always have a finite efficiency, this efficiency is high in most practical cases, and the influence of the nonlinear behavior of the isotherm becomes the controlling factor of band profiles, at least at high concentrations [7]. As shown later, a concentration band may migrate in two opposite directions and exit at two opposite side outlets of the column; we define the raffinate outlet at which the component exits with the liquid phase (after moving in the direction of the liquid phase) and the extract outlet at which the component exits with the solid phase.

2.1. System of equations of countercurrent chromatography

Based on the above assumptions, the mass balance of a retained component is given by

$$\frac{\partial C}{\partial t} + F \cdot \frac{\partial q}{\partial t} + u \cdot \frac{\partial C}{\partial z} - vF \cdot \frac{\partial q}{\partial z} = 0 \qquad (1)$$

where C and q are the liquid- and solid-phase concentrations of the component, respectively, t and z are the time and the position in the column, respectively, F is the phase ratio [$F = (1-\varepsilon)/\varepsilon$, ε = total column porosity], u is the flow velocity of the liquid phase and v is the velocity of the moving solid phase. Eq. 1 differs from the conventional mass balance of chromatography only by the second convective term.

This equation is also the mass balance in simulated moving-bed chromatography, provided that the velocity u and v are chosen so that the following two relationships are valid:

$$v = v^{SMB} \qquad (2a)$$

$$u + Fv = u^{SMB} \qquad (2b)$$

where u^{SMB} and v^{SMB} are the velocities of the fluid (liquid or gas) and the solid phases, respectively, in SMB. Since, in practice, the movement of the solid phase in SMB is achieved by switching sequentially the inlet and outlet ports of each of the series of interconnected sub-columns, the velocity v^{SMB} is such that

$$v^{SMB} = \frac{L_s(1-\varepsilon)}{t_s} \qquad (2c)$$

where L_s is the length of each sub-column or column sub-section used in SMB and t_s is the switching interval time, i.e., the time between two successive shifts of the feed injection and product withdrawal points, shifts which are assumed to take place instantaneously.

C and q are related by the adsorption isotherm:

$$q = f(C) \qquad (3a)$$

In this work, q is assumed to be differentiable as needed. For the sake of simplicity we consider only a convex-upward isotherm without an inflection point. The extension to a convex-downward isotherm is straightforward and is not discussed here. The simplest and most often used model, the Langmuir isotherm:

$$q = \frac{aC}{1+bC} \qquad (3b)$$

where a and b are numerical coefficients, will be used for illustrative examples of our general results.

The initial and boundary conditions correspond to an empty column and to the injection of a rectangular pulse of feed in the middle of the column, respectively. This amendment to the classical definition of the boundary condition is required to account for all the possible movements of the band. As shown later, depending on the experimental conditions, the band (or part of it) can move in the forward or backward direction. Feed injection in the middle of the column differs from conventional practice in

analytical chromatography or in fixed-bed preparative chromatography, in which cases the injection is done at one end of the column. Actually, this condition corresponds to the practice of countercurrent moving-bed chromatography where the feed is introduced at the column center and the separated fractions are collected at its two ends. The initial and boundary conditions are written as

$$C(z, t=0) = 0 \qquad -L < z < L \tag{4a}$$

$$C(z=0, t) = C_0 \qquad 0 < t < t_p \tag{4b}$$

$$C(z=0, t) = 0 \qquad t_p < t \tag{4c}$$

where the initial component concentration in the column is zero, C_0 is the component concentration in the feed and t_p is the injection duration.

2.2. Summary of the results obtained in fixed-bed chromatography

The properties of the band profiles predicted by the ideal model have been abundantly discussed in the case of fixed-bed chromatography [7–16]. In this case, $v = 0$ and the mass balance equation contains only the first three terms in Eq. 1. It results from this equation that a velocity $u_z(C)$ is associated with each concentration C on a diffuse (i.e., continuous) profile. This velocity is given by

$$u_z = \frac{u}{1 + F \cdot \dfrac{dq}{dC}} \tag{5}$$

Thus, in the case of a rectangular pulse injection of width t_p and height C_0, the continuous part of the elution profile (its rear in the case of a convex-downward isotherm such as the Langmuir isotherm) is given by

$$t(C) = t_p + \frac{L}{u_z} = t_p + t_0\left(1 + F \cdot \frac{dq}{dC}\right) \tag{6}$$

Because the isotherm is not linear, u_z depends on the concentration. As a consequence, a concentration discontinuity or shock will form on one side of the profile, whatever the injection profile. In Eq. 5, dq/dC is the slope of the isotherm at concentration C. It increases with decreasing liquid-phase concentration for a convex-upward isotherm, with increasing concentration for a convex-downward isotherm. Thus, the velocity associated with a concentration increases with increasing concentration in the case of a convex-upward isotherm. Fig. 1 illustrates this effect for the Langmuir isotherm (solid line). In this case, high concentrations move faster than low concentrations but, as they cannot pass them [7,10,11], they pile up at the front of the band and a discontinuity or shock is formed. The opposite is true for a convex-downward isotherm, in which case a concentration shock forms at the rear of the band profile.

The velocity of the concentration shock is not given by Eq. 5. Writing a mass balance for the discontinuity, Aris and Amundson [10] have shown that the shock velocity is given by

$$U_s = \frac{u}{1 + F \cdot \dfrac{\Delta q}{\Delta C}} \tag{7}$$

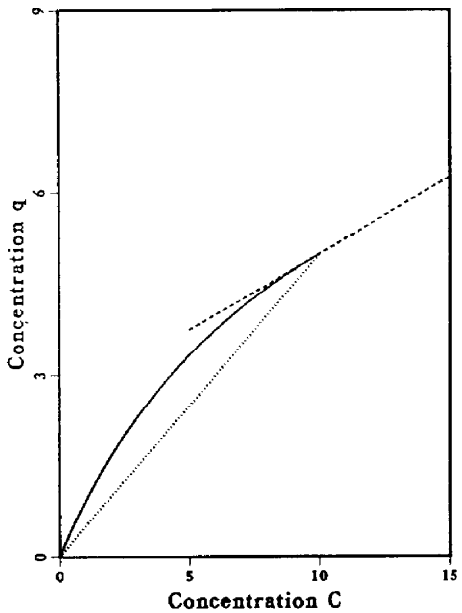

Fig. 1. Langmuir isotherm. Solid line, isotherm; dotted line, tangent to the isotherm at C; dashed line, chord of the isotherm at C. The parameters used in all the figures are phase ratio, $F = 0.25$; injection time, $t_p = 0.5$ s; Langmuir isotherm, $a = 1$ and $b = 0.1$ mM^{-1}. Axes in mM.

where Δq and ΔC are the concentration amplitudes of the shock in the solid and the liquid phase, respectively. The ratio $\Delta q/\Delta C$ is the slope of the chord connecting the two points on the isotherm which correspond to the liquid-phase composition after and before the shock (see Fig. 1). Unless the isotherm has an inflection point, this line goes through the origin.

The retention time of the front shock can be obtained by writing that the band area is constant and equal to the area injected [7,16]. This procedure provides the maximum concentration of the band, given as the root of the algebraic equation

$$\left| q - C \cdot \frac{dq}{dC} \right| = \frac{n}{F_v t_0 F} \tag{8}$$

where $n = C_0 t_p F_v$ is the amount injected, $F_v = \varepsilon S u$ is the liquid-phase flow-rate and S is the column cross-sectional area. This equation can be solved in the case of the Langmuir isotherm, giving

$$C_M = \frac{\sqrt{L_f}}{b(1 - \sqrt{L_f})} \tag{9}$$

where L_f is the loading factor or ratio of the sample size to the column saturation capacity:

$$L_f = \frac{n}{F_v t_0 F} = \frac{t_p C_0 b}{t_0 k'_0} \tag{10}$$

where $k'_0 = Fa$ is the limit retention factor at infinite dilution. In the case of the Langmuir isotherm, the retention time, $t_R(C_M)$, of the shock is given by

$$t_R(C_M) = t_p + t_0 + k'_0 t_0 (1 - \sqrt{L_f})^2 \tag{11}$$

A band profile corresponding to this case is shown in Figs. 2 and 3 (solid lines), as a reference for comparison with the profiles obtained in the case of a moving-bed column.

Finally, comparison between Eqs. 5 and 7 shows that the point on a continuous concentration profile, or diffuse boundary, at concentration C moves faster than a shock from concentration 0 to C. Hence, when a rectangular pulse (width t_p, height C_0) is injected into the column, its front shock is stable and moves as a shock at a constant velocity [Eq. 7, $U_s(C_0)$]. Its rear shock is not stable and becomes a diffuse boundary (Eq. 6). The highest point of that

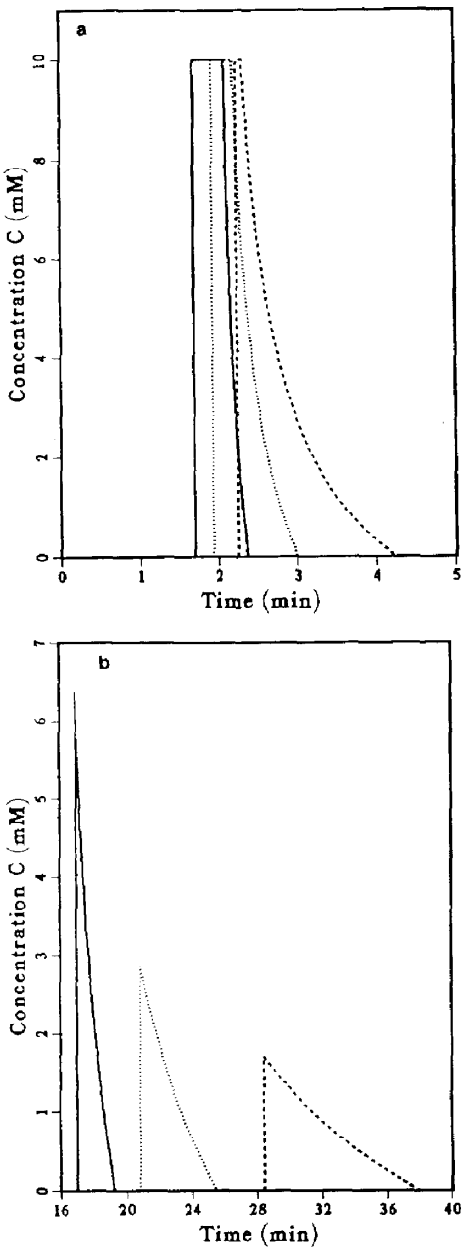

Fig. 2. Positive velocity case. Band profiles at the raffinate outlet. $k'_0 = 0.25$ Solid line, fixed-bed chromatography, $\beta = 0$; dotted line, moving-bed chromatography, $\beta = 1$; dashed line, moving-bed chromatography, $\beta = 2$. Liquid-phase flow velocity, $u = 6.67$ cm/s. Column length: (a) 10 cm; (b) 100 cm.

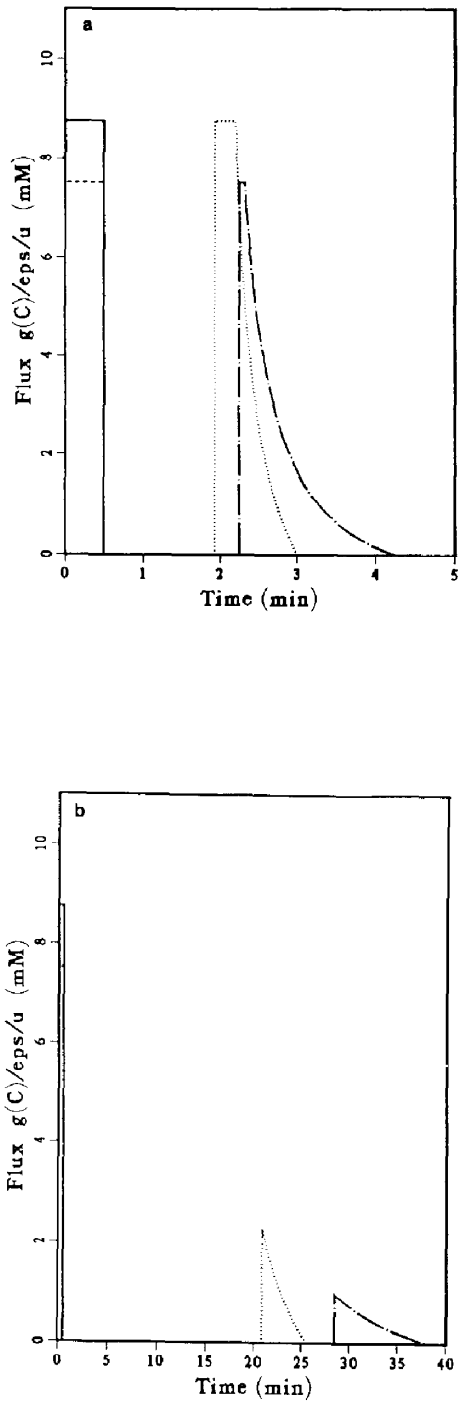

Fig. 3. Positive velocity case. Mass fluxes at the raffinate outlet. Solid line, flux profile at injection or in-flux for $\beta = 1$; dotted line, out-flux profile for $\beta = 1$; dashed line, in-flux profile for $\beta = 2$; chain-dotted line, out-flux profile for $\beta = 2$; liquid-phase flow velocity, $u = 6.67$ cm/s. Column length: (a) 10 cm; (b) 100 cm. eps = ε.

boundary moves faster than the shock, so the injection plateau at C_0 shrinks and eventually disappears. When it disappears, the shock amplitude decreases and so does its velocity (Eq. 7). The shock height is given by Eq. 8 and its position by Eq. 6 as $t_R(C_M)$.

Similar profiles are obtained in countercurrent chromatography, with some significant changes caused by the solid-phase migration.

2.3. General properties of band profiles in countercurrent chromatography

In this section we follow the same approach as in fixed-bed chromatography to derive the equations giving the features of the band profiles during their migration along the column [$C = f(z)$ at t = constant] and those of the elution profiles [$C = f(t)$ at $z = L$ or $z = -L$ according to the band exiting at the raffinate outlet or the extract outlet].

Mass conservation and mass flux

During a classical chromatographic experiment, the concentration of a component is a function of both t and z. There are significant differences in fixed-bed chromatography between elution profiles [i.e., curves $C(t)$ at $z = L$] and concentration profiles along the column or spatial profiles [i.e., curves $C(z)$ at a certain time]. In the latter case, the component in the column is distributed between the two phases and it is the sum of the areas of the liquid- and solid-phase concentration profiles which is conserved. Because of the non-linear behavior of the isotherm and the progressive dilution of the band during its migration, the proportion of the component in each phase at equilibrium varies during elution. The proportion in the solid phase increases during the migration of the band if the isotherm is convex-upward because the average concentration decreases. In the elution profile, however, the entire amount of component is in the liquid phase, so the area of the elution profile is equal to the area of the injection profile. In moving-bed chromatography, by contrast, the history of liquid-phase concentrations at a given point of the column (e.g., at $z = L$) does not have a constant area. Part of the

component moves past the point z with the liquid phase at velocity u, and part with the solid phase at velocity v. The invariant quantity in this case is the integral of the mass flux passing across the column at point z. It must be equal to the total amount injected.

In fixed-bed chromatography, the net mass flux, $g^0(C)$, in any point of the column for a component with a local concentration C is

$$g^0(C) = \varepsilon C u \tag{12}$$

As εu is constant, the mass flux is proportional to the concentration and the area of the liquid-phase concentration profile is proportional to the mass and can represent it.

In a countercurrent moving bed, the solid phase moves in the countercurrent direction and the net flux, $g^\beta(C)$, is

$$g^\beta(C) = \varepsilon C u - (1-\varepsilon) q v = \varepsilon u (C - \beta F q) \tag{13}$$

where $\beta = v/u$ is the ratio of the velocities of the solid and liquid phases in the moving bed column. It is the area of the profile of $g^\beta(C)$ which is constant at any point of the column and, for practical purposes, at $z = 0$ (inlet of the liquid phase and feed in the present case) and at $z = L$ or $z = -L$ (outlets of the liquid phase and component).

Velocity associated with a concentration

In the cases of MB or SMB, Eq. 1 can be rewritten as

$$\frac{\partial C}{\partial t} + u\left(\frac{1 - \beta F \cdot \frac{dq}{dC}}{1 + F \cdot \frac{dq}{dC}}\right)\frac{\partial C}{\partial z} = 0 \tag{14}$$

This result is very similar to that obtained in elution on a fixed-bed column (Eq. 5), a case in which $\beta = v = 0$. Eq. 14 shows that each concentration propagates in the column at the velocity u_z given by

$$u_z = u\left(\frac{1 - \beta F \cdot \frac{dq}{dC}}{1 + F \cdot \frac{dq}{dC}}\right) \overset{\text{Langmuir}}{=} u\left[1 - \frac{k'_0(1+\beta)}{(1+bC)^2 + k'_0}\right] \tag{15}$$

Properties similar to those of the elution profiles on a fixed-bed column have been found for the velocity associated with a concentration on a diffuse boundary and for the velocity of the concentration shock in moving-bed chromatography [3,11]. Of special importance, u_z increases with increasing concentration for a convex-upward isotherm. In this case, accordingly, the bands will have a rear diffuse boundary and a front shock, as in the fixed-bed case.

However, in the moving-bed case the velocity associated with a concentration on a diffuse boundary (Eq. 15) may be positive or negative, depending on the sign of $(1 - \beta F\, dq/dC)$. We need to consider separately these two possible cases. In fact, in the most common applications of MB and SMB implementations, the feed will be separated into two fractions on the basis of the sign of the velocities associated with low concentrations of the two pure components. The simplest practical case is the separation of a binary mixture, with one component moving in one direction and the other in the opposite direction.

Shock velocity and mass propagation

When a concentration shock is formed, its velocity is not given by Eq. 15, but by an equation similar to Eq. 7 and derived in the same manner:

$$U_s = u\left(\frac{1 - \beta F \cdot \frac{\Delta q}{\Delta C}}{1 + F \cdot \frac{\Delta q}{\Delta C}}\right) \overset{\text{Langmuir}}{=} u\left[1 - \frac{k'_0(1+\beta)}{(1+bC) + k'_0}\right] \tag{16}$$

where $\Delta q/\Delta C$ is the slope of the isotherm chord (Fig. 1). Comparison between Eqs. 15 and 16 shows that for a given concentration C, u_z will be larger than U_s in the case of a convex-upward isotherm, as it is in fixed-bed chromatography. Hence the concentration plateau obtained in the case of a rectangular pulse injection (boundary conditions 4b and 4c) shrinks and disappears, leaving a concentration shock of decreasing amplitude which moves at a decreasing velocity.

In fixed-bed chromatography, the migration

velocity of a slice of thickness dz of the profile of a retained component is given by

$$V^0 = \frac{\text{net flux}}{\text{total mass}} = \frac{\varepsilon C u}{\varepsilon C + (1-\varepsilon)q} = \frac{u}{1+\frac{Fq}{C}} \overset{\text{Langmuir}}{=} \frac{u}{1+\frac{k_0'}{1+bC}} \quad (17)$$

Thus, as explained and illustrated elsewhere (Ref. [5], pp. 227–228), the mass of component contained in a slice of column of thickness dz where the concentration is C migrates at the velocity of the shock, U_s (Eq. 16), while the concentration C moves at the velocity u_z. Similarly, in moving-bed chromatography, the migration velocity of a slice of thickness dz of the profile of a retained component is

$$V^\beta = \frac{\text{net flux}}{\text{total mass}} = \frac{\varepsilon C u - (1-\varepsilon)qv}{\varepsilon C + (1-\varepsilon)q} \overset{\text{Langmuir}}{=} u\left(\frac{1-\frac{\beta k_0'}{1+bC}}{1+\frac{k_0'}{1+bC}}\right) \quad (18)$$

In moving-bed chromatography also the mass contained in a column slice of thickness dz propagates at the shock velocity, while concentrations propagate at the faster velocity u_z. However, whereas in fixed-bed chromatography V^0 is always positive and a retained component always moves forward, in moving-bed chromatography, by contrast, the mass velocity, V^β, is proportional to $(1-\beta Fq/C)$ which depends on the velocity ratio, β, and on the local solid- and liquid-phase concentrations, q and C. V^β may be positive, negative or zero.

2.4. Equations for the band profile in the different possible cases

The injection profile is described by Eqs. 4b and 4c. It includes two concentration shocks. The first, at the injection front, enters the column at time $t = 0$; the second, at the injection rear, enters the column at time $t = t_p$. Between these two shocks, there is a concentration plateau at $C = C_0$ with a width t_p. Because the band may change direction of propagation when the shock amplitude decreases (see later), we need to clarify the concepts of the front and rear of the band. We define the band front as the part of the profile which arises from the front shock of the injection pulse, that which enters the column first; the band rear is the part of the profile arising from the rear shock of the rectangular injection front. Owing to the complexity of countercurrent chromatography, the front so defined may be eluted last, which should not be surprising as it may leave the column through the extract exit.

The solution requires now that we distinguish several possibilities, depending on the sign of the velocities associated with the concentration shocks and with the concentration $C = 0$. If the shock velocity is positive, it begins its migration in the direction of the liquid phase. If it is negative, it migrates in the direction of the solid phase. However, if the shock velocity is positive, the velocity of the concentration $C = 0$ can be positive or negative. In the former case, the entire band will eventually elute with the liquid phase. In the latter case, the shock, which migrates in the direction of the liquid phase but erodes progressively and slows, may or may not reach the liquid phase exit. We discuss all these cases in detail.

First case: all concentration velocities are positive, $\beta k_0' < 1$

In this case, all the concentrations migrate in the direction of the liquid phase. Since for a convex-upward isotherm, dq/dC decreases with increasing concentration, C, the associated velocity (Eq. 15) increases with increasing concentration. Because high concentrations move faster than lower concentrations but cannot pass them, the front shock is stable and will propagate along the column. In contrast, the rear shock is unstable, collapses in a flight of characteristics and spreads continuously [7,10,11].

Since the rear shock becomes a diffuse boundary, each concentration C moves at the velocity given by Eq. 15. The retention time of concentration C at the column outlet is

$$t_R^\beta(C) = t_p + \frac{L}{u_z} = t_p + t_0\left(\frac{1 + F\cdot\frac{dq}{dC}}{1 - \beta F\cdot\frac{dq}{dC}}\right)$$

$$\overset{\text{Langmuir}}{=} t_p + t_0\left[1 + \frac{(1+\beta)k_0'}{(1+bC)^2 - \beta k_0'}\right] \qquad (19a)$$

where t_p results from the injection duration and the fact that the rear diffuse boundary results from the collapse of the rear shock of the injection profile. Since dq/dC decreases with increasing concentration, $t_R^\beta(C)$ also decreases with increasing C. Eq. 19 shows that $t_R^\beta(C)$ increases with increasing β, corresponding to an increase in the effective column length seen by the band which moves against the direction of the solid phase.

The profile ends at the elution time of the concentration $C = 0$:

$$t_R^\beta(0) = t_p + t_0\cdot\frac{1 + k_0'}{1 - \beta k_0'} \qquad (19b)$$

with $k_0' = F[dq/dC]_0$, where $[dq/dC]_0$ is the initial slope of the isotherm.

The front shock of the injection pulse moves at the constant velocity $U_s(C_0)$ as long as the injection plateau is not eroded away (see below). If the injection plateau still exits, the retention time of the shock at the raffinate exit $z = L$ is

$$t_s^\beta(C_0) = \frac{L}{U_s(C_0)} = t_0\left(\frac{1 + F\cdot\frac{\Delta q}{\Delta C}}{1 - \beta F\cdot\frac{\Delta q}{\Delta C}}\right)$$

$$\overset{\text{Langmuir}}{=} t_0\left[1 + \frac{(1+\beta)k_0'}{(1+bC_0) - \beta k_0'}\right] \qquad (20)$$

where $t_0 = L/u$ is the holdup time and $\Delta q/\Delta C = [q(C_0) - q(0)]/C_0$ is the slope of the isotherm chord, as illustrated by the dotted line in Fig. 1 (in the case of the Langmuir isotherm).

Since the rear of the band profile is a diffuse boundary whereas its front is a shock, the two ends of the injection plateau propagate at different velocities. The velocity of the front shock is given by Eq. 16 and its retention time is given by Eq. 20. All concentrations on the rear diffuse boundary move at velocities given by Eq. 15, and their retention times are given by Eq. 19. The point at the rear of the plateau is the highest point of the rear diffuse boundary, with a retention time $t_R^\beta(C_0)$ derived from Eq. 19 with $C = C_0$. For the convex-upward isotherm considered here (see Fig. 1), $dq/dC(C_0) < \Delta q(C_0)/\Delta C = [q(C_0) - q(0)]/C_0$ and $u_z(C_0) > U_s(C_0)$. Hence the velocity of the point of the diffuse profile at $C = C_0$ is larger than the shock velocity. The plateau becomes narrower, shrinks and eventually disappears if the column is long enough. Then, the rear diffuse boundary captures the front, the shock erodes, its height decreases and it slows (see Eq. 16). The tip of the band profile belongs both to the front shock and to the rear diffuse boundary. u_z is larger than U_s for any given value of C. As a consequence, the point at the tip of the shock disappears constantly and the shock keeps becoming shorter and moving more slowly. Therefore, after the concentration plateau at $C = C_0$ has disappeared, Eq. 16 can no longer be used to calculate the retention time of the shock.

This time is easily derived by observing that the total mass in the column must be conserved and that, accordingly, the area of the mass flux at the raffinate end (in the present case) of the column, $z = L$, must be equal to the area of the injection profile [16]. We can calculate the net mass of component leaving the column by integrating the net mass flux profile of the component (Eq. 13) from $C = 0$ to $C = C_M$, where C_M is the maximum concentration of the profile. This gives

$$M^{out} = \int_0^{C_M} [t_R^\beta(C) - t_R^\beta(C_M)]\, dg^\beta(C)$$

$$= t_0\varepsilon u\left\{[C + Fq(C)]\Big|_0^{C_M} - \left(\frac{1 + F\cdot\frac{\Delta q}{\Delta C}}{1 - \beta F\cdot\frac{\Delta q}{\Delta C}}\right) g^\beta(C)\Big|_0^{C_M}\right\} \qquad (21a)$$

The mass flux entering the column is given by the equation

$$M^{in} = \int_0^{C_0} t_p \, dg^{\beta}(C) = t_p g^{\beta}(C)|_0^{C_0} \quad (21b)$$

Writing that these two masses are equal gives

$$t_p[C - \beta F q(C)]|_0^{C_0} = t_0 \left\{ [C + Fq(C)]|_0^{C_M} - \frac{1 + F \cdot \frac{dq(C_M)}{dC}}{1 - \beta F \cdot \frac{dq(C_M)}{dC}} [C - \beta F q(C)]|_0^{C_M} \right\} \quad (21c)$$

C_M is obtained as the root of this algebraic equation. Once it is known, its retention time is derived from Eq. 19 as $t_R^{\beta}(C_M)$.

The above derivations apply to any isotherm, with the only condition that it has no inflection point. In the case of the Langmuir isotherm (Eq. 21c), C_M can be solved in closed form and the maximum concentration of the band is

$$C_M = \frac{L_f^{\beta} + \sqrt{L_f^{\beta}(1 + (L_f^{\beta} - 1)\beta k_0')}}{b(1 - L_f^{\beta})} \quad (22)$$

The retention time of the maximum concentration of the band when the injection plateau has been eroded away is given by

$$t_R^{\beta}(C_M) = t_p + t_0 + t_0(1 + \beta)k_0' \times \frac{(1 - L_f^{\beta})^2}{L_f^{\beta}(1 + \beta k_0') + 1 - \beta k_0' + 2\sqrt{L_f^{\beta}(1 + L_f^{\beta}\beta k_0' - \beta k_0')}} \quad (23)$$

where L_f^{β} is the loading factor, or ratio of the sample size and the column saturation capacity:

$$L_f^{\beta} = \frac{\left(1 - \frac{\beta k_0'}{1 + bC_0}\right) t_p C_0 b}{t_0 k_0'(1 + \beta)} \quad (24)$$

The loading factor is lower in moving-bed chromatography than with a fixed bed of the same length because moving the solid phase backwards effectively increases the length of the column bed along which a component must migrate before eluting out of the column.

Obviously, the solution of the moving-bed problem in the case in which the concentration velocities are all positive is very similar to the solution of fixed-bed chromatography. A comparison of the relevant equations shows that the equations giving the band profile in moving-bed chromatography tend toward those derived for fixed-bed chromatography when the velocity of the solid phase tends toward 0 (see Table 1). A discussion of this comparison follows. It is convenient for the purpose of this discussion to report the retention time of the shock and the elution times of concentrations with respect to those observed under fixed-bed conditions. These expressions can be written explicitly in the case of Langmuir isotherm.

Retention times in fixed-bed and moving-bed chromatography. Assuming that all the experimental conditions are the same for the moving and the fixed beds, except for the movement of the solid phase in the former case, the retention times of a shock of amplitude C and of a concentration C become

$$\tau_s^{\beta}(C) = \frac{t_s^{\beta}(C) - t_s^0(C)}{t_s^0(C)} = \frac{1}{\frac{1}{\beta F \cdot \frac{\Delta q}{\Delta C}} - 1} \overset{\text{Langmuir}}{=} \frac{1}{\frac{1 + bC}{\beta k_0'} - 1} \quad (25a)$$

$$\tau_R^{\beta}(C) = \frac{t_R^{\beta}(C) - t_R^0(C)}{t_R^0(C)} = \frac{1}{\frac{1}{\beta F \cdot \frac{dq}{dC}} - 1} \overset{\text{Langmuir}}{=} \frac{1}{\frac{(1 + bC)^2}{\beta k_0'} - 1} \quad (25b)$$

where $\tau_s^{\beta}(C)$ and $\tau_R^{\beta}(C)$ are the relative differences of the retention times of the front shock of amplitude C and of a concentration C, respectively, between moving-bed and fixed-bed columns. Since in the present case $\beta k_0' < 1$ and $d^2q/dC^2 < 0$ (convex-upward isotherm), we have always $\beta F \, dq/dC < 1$ and, as a consequence, τ_s^{β} and τ_R^{β} are positive. They increase with increas-

Table 1

Moving-bed chromatography	Fixed-bed chromatography
General isotherm model, $q = f(C)$	
$t_R^\beta(C) = t_p + t_0\left(\frac{1 + F\cdot\frac{dq}{dC}}{1 - \beta F\cdot\frac{dq}{dC}}\right)$	$t_R^0(C) = t_p + t_0\left(1 + F\cdot\frac{dq}{dC}\right)$
$t_s^\beta(C_0) = t_0\left(\frac{C_0 + Fq_0}{C_0 - \beta F q_0}\right)$	$t_s^0(C_0) = t_0\left(1 + F\cdot\frac{q_0}{C_0}\right)$
$t_p[C - \beta F q(C)]_0^{C_0} = t_0\left\{[C + Fq(C)]_0^{C_M} - \frac{1 + F\cdot\frac{dq(C_M)}{dC}}{1 - \beta F\cdot\frac{dq}{dC}(C_M)}[C - \beta F q(C)]_0^{C_M}\right\}$	$\left\lvert q - C\cdot\frac{dq}{dC}\right\rvert = \frac{n}{F_v t_0 F}$
Case of the Langmuir isotherm, $q = aC/(1 + bC)$	
$t_R^\beta(C) = t_p + t_0\left[1 + \frac{(1+\beta)k_0'}{(1+bC)^2 - \beta k_0'}\right]$	$t_R^0(C) = t_p + t_0\left[1 + \frac{k_0'}{(1+bC)^2}\right]$
$t_s^\beta(C_0) = t_0\left[1 + \frac{(1+\beta)k_0'}{(1+bC_0) - \beta k_0'}\right]$	$t_s^0(C_0) = t_0\left(1 + \frac{k_0'}{1+bC_0}\right)$
$t_R^\beta(C_M) = t_p + t_0 + t_0(1+\beta)k_0'\cdot\frac{(1 - L_f^\beta)^2}{L_f^\beta(1+\beta k_0') + 1 - \beta k_0' + 2\sqrt{L_f^\beta(1 + L_f^\beta\beta k_0' - \beta k_0')}}$	$t_R^0(C_M) = t_p + t_0 + k_0' t_0(1 - \sqrt{L_f})^2$
$L_f^\beta = \frac{\left(1 - \frac{\beta k_0'}{1+bC_0}\right)t_p C_0 b}{t_0 k_0'(1+\beta)}$	$L_f^0 = \frac{nb}{\varepsilon S L k_0'} = \frac{t_p C_0 b}{t_0 k_0'}$

ing β and with decreasing concentration of the component in the liquid phase.

This result is expected. The migration of the solid phase in the direction opposite to the liquid-phase stream and to the direction in which the lesser retained bands migrate increases the apparent column length and the actual migration distance of a band in the column. The larger is β, the longer are the retention times. This effect is illustrated in Fig. 2a and b, which show the band profiles obtained at the end of a short (Fig. 2a, $L = 10$ cm) and a long column (Fig. 2b, $L = 100$ cm), for the same rectangular injection ($C_0 = 10$ mM) and the same velocity of the solid phase. The injection pulse width has been chosen so that the band elutes from the short column before the injection plateau has been eroded away.

Erosion of the injection plateau between the shock and the diffuse boundary. The width of the injection plateau decreases continuously because the highest point on the diffuse boundary moves faster than the shock. The width of this plateau is given by

$$t_R^0(C_0) - t_s^0(C_0) = t_p + t_0\left[\frac{dq}{dC}(C_0) - \frac{\Delta q}{\Delta C}\right] \quad (26a)$$

in the fixed-bed case. In the moving-bed case, it is

$$t_R^\beta(C_0) - t_s^\beta(C_0) = t_p + t_0 \times \left\{\frac{(1+\beta)F\left[\frac{dq}{dC}(C_0) - \frac{\Delta q}{\Delta C}\right]}{\left[1 - \beta F\cdot\frac{dq}{dC}(C_0)\right]\left(1 - \beta F\cdot\frac{\Delta q}{\Delta C}\right)}\right\} \quad (26b)$$

As $\beta\, dq/dC < 1$ and $dq/dC(C_0) < \Delta q/\Delta C(C_0)$, we have

$$t_R^\beta(C_0) - t_s^\beta(C_0) < t_R^0(C_0) - t_s^0(C_0) \quad (26c)$$

The width of the residual injection plateau at the top of the band profile is smaller in the moving-

bed case than in the corresponding fixed-bed case. This is clearly illustrated by the chromatograms calculated in Fig. 2a.

Profile of the diffuse boundary. Both τ_R^β and t_R^0 decrease with increasing liquid phase concentration; so also does $t_R^\beta(C) - t_R^0(C)$, and

$$t_R^\beta(C_0) - t_R^0(C_0) \leq t_R^\beta(C) - t_R^0(C) \leq t_R^\beta(0) - t_R^0(0) \quad (27)$$

This means that the retention time of a given concentration, which increases with decreasing concentration, increases faster in moving-bed than in fixed-bed chromatography. The band width or distance between the band front or shock and its point at any concentration C is larger under moving-bed than under fixed-bed conditions. In other words, the rear diffuse boundary is more diffuse for the moving-bed than for the fixed-bed column and its width increases with increasing β or solid-phase velocity. This effect is illustrated in Fig. 2a and b.

As a consequence, the elution bands are more dilute in moving-bed than in fixed-bed chromatography. This conclusion has no direct consequence, however, on the concentration of the fractions collected in the conventional applications of the method.

Band profile area. In fixed-bed chromatography, the area of the elution band profile represents the mass leaving the column, as explained previously. Because the component mass in the column must be conserved, this area must remain constant and equal to the area of the injection profile. This property is used to determine the maximum concentration of the band, once the injection plateau has been eroded away (see Theory). Inside the column, however neither the area of the liquid-phase nor that of the solid-phase concentration profile remains constant. It is the total concentration $(C + Fq)$ profile area that remains constant, whether in fixed-bed or in moving-bed chromatography. This is due to the non-linear behavior of the isotherm which causes a variable proportion of the component to be in each phase at equilibrium.

It is interesting to calculate the area of the liquid-phase concentration profile at the column exit and compare it with the injection area. The area of a rectangular injection pulse is

$$S^{in} = t_p C_0 \quad (28a)$$

The band area at the outlet of column in fixed-bed chromatography is

$$S^{out} = \int_0^{C_0} [t_R^0(C) - t_s^0(C_0)]\, dC = t_p C_0 + t_0 F \int_0^{C_0} \left(\frac{dq}{dC} - \frac{\Delta q}{\Delta C}\right) dC = S^{in} \quad (28b)$$

This result is obtained because $dq|_0^{C_0} = (\Delta q/\Delta C)\Delta C$. This demonstration is general, independent of the isotherm properties, whether convex upward or downward. It results from the fact that the profile is a solution of the mass balance equation and that only the liquid phase leaves the column.

In moving-bed chromatography, the area of the elution profile of the liquid-phase concentration band is

$$S^{out} = \int_0^{C_0} [t_R^0(C) - t_s^0(C_0)]\, dC = t_p C_0 + t_0 F \int_0^{C_0} \left\{ \frac{(1+\beta)\left[\frac{dq(C)}{dC} - \frac{\Delta q}{\Delta C}\right]}{\left[1 - \beta F \cdot \frac{dq(C)}{dC}\right]\left(1 - \beta F \cdot \frac{\Delta q}{\Delta C}\right)} \right\} dC \quad (29a)$$

We have no solution for Eq. 29. However, in the case of the Langmuir isotherm, it simplifies to

$$S^{out} - S^{in} = t_0(1+\beta)k_0' \left\{ \frac{1}{2b\gamma} \ln\left[\frac{(1 + bC_M - \gamma)(1+\gamma)}{(1 + bC_M + \gamma)(1-\gamma)} - \frac{C_M}{(1 + bC_M) - \gamma^2} \right] \right\} > 0 \quad (29b)$$

with $\gamma = \sqrt{\beta k_0'}$. The right-hand side of Eq. 29b is always positive for the case studied here, since it

is equal to 0 when $C_M = 0$ and its first differential with respect to C_M is positive. Thus, the area of the band profile of the liquid-phase concentration is not conserved but is larger at the column outlet than at injection. It increases with increasing β. This reflects the fact that the concentration of the component in the solid phase that enters the column (and is in equilibrium with the exiting liquid phase) is finite and that this phase and the liquid phase move in opposite directions. Hence part of the amount eluted in the liquid phase returns into the column with the solid phase.

As an example, while the areas of the injection profiles of the short and long columns in Fig. 2a and b are identical (dotted line, $\beta = 1$), the areas of the profiles of the liquid-phase concentration obtained at the outlet of each of these two columns are 1.032 (Fig. 2a) and 1.16 (Fig. 2b), respectively, larger for moving-bed than for fixed-bed operation. The problem of the mass conservation is easily solved by considering the local mass flux.

Mass flux and mass conservation. Based on the net mass flux of component given in Eq. 13, we can calculate the masses injected into the column and leaving the column in the case in which a residual injection plateau is included in the elution profile

$$M^{in} = \int_0^{C_0} t_p \, dg^{\alpha}(C) = t_p g^{\alpha}(C)|_0^{C_0} \tag{30a}$$

$$M^{out} = \int_0^{C_0} [t_R^{\alpha}(C) - t_s^{\alpha}(C_0)] \, dg^{\alpha}(C) \tag{30b}$$

$$= \int_0^{C_0} \left[t_p + t_0 \left(\frac{1 + F \cdot \frac{dq}{dC}}{1 - \alpha F \cdot \frac{dq}{dC}} \right) - t_0 \left(\frac{1 + F \cdot \frac{\Delta q}{\Delta C}}{1 - \alpha F \cdot \frac{\Delta q}{\Delta C}} \right) \right] dg^{\alpha}(C)$$

$$= t_p g^{\alpha}(C)|_0^{C_0} + t_0 \varepsilon u \left\{ [C + Fq(C)]|_0^{C_0} - \left(\frac{1 + F \cdot \frac{\Delta q}{\Delta C}}{1 - \alpha F \cdot \frac{\Delta q}{\Delta C}} \right) g^{\alpha}(C)|_0^{C_0} \right\}$$

$$= t_p g^{\alpha}(C)|_0^{C_0} = M^{in} \tag{30c}$$

Mass is indeed conserved, which is expected in the case of the exact solution of a mass balance equation. A similar calculation was used to determine the maximum concentration of the band, and hence its retention time when the shock has been captured by the diffuse boundary, assuming conservation of the net mass flux (Eq. 21). However, it was important to illustrate the difference between fixed-bed and moving-bed problems and to emphasize that, since the solid phase moves in and out the column as well as along it and carries a fraction of the retained component, its contribution to the solute mass flux must be taken into account in the latter case.

Fig. 3 illustrates the time profiles of the in-flux and out-flux for (a) the short and (b) the long column, with the rectangular profiles showing the injection or in-flux profiles and the curved lines showing the out-flux in the case for which $\beta = 1$ (dashed line) and $\beta = 2$ (chain-dotted line). The areas of the in-flux and out-flux profiles are now the same, but note that, although the concentration profiles at injection are the same for the two values of β, the in-flux profiles are different (see Eq. 13).

Second case: the limit velocity is zero, $\beta k_0' = 1$

As in the previous case, any finite concentration migrates at a positive velocity, i.e., in the direction of the liquid phase, but the limit velocity, or velocity associated with the concentration $C = 0$, is zero. This case differs from the first case, in which even the limit velocity is strictly positive (i.e., different from zero). Now the end of the band ($C = 0$) does not move at all, as illustrated in Fig. 4a, which shows concentration profiles along the column. All these

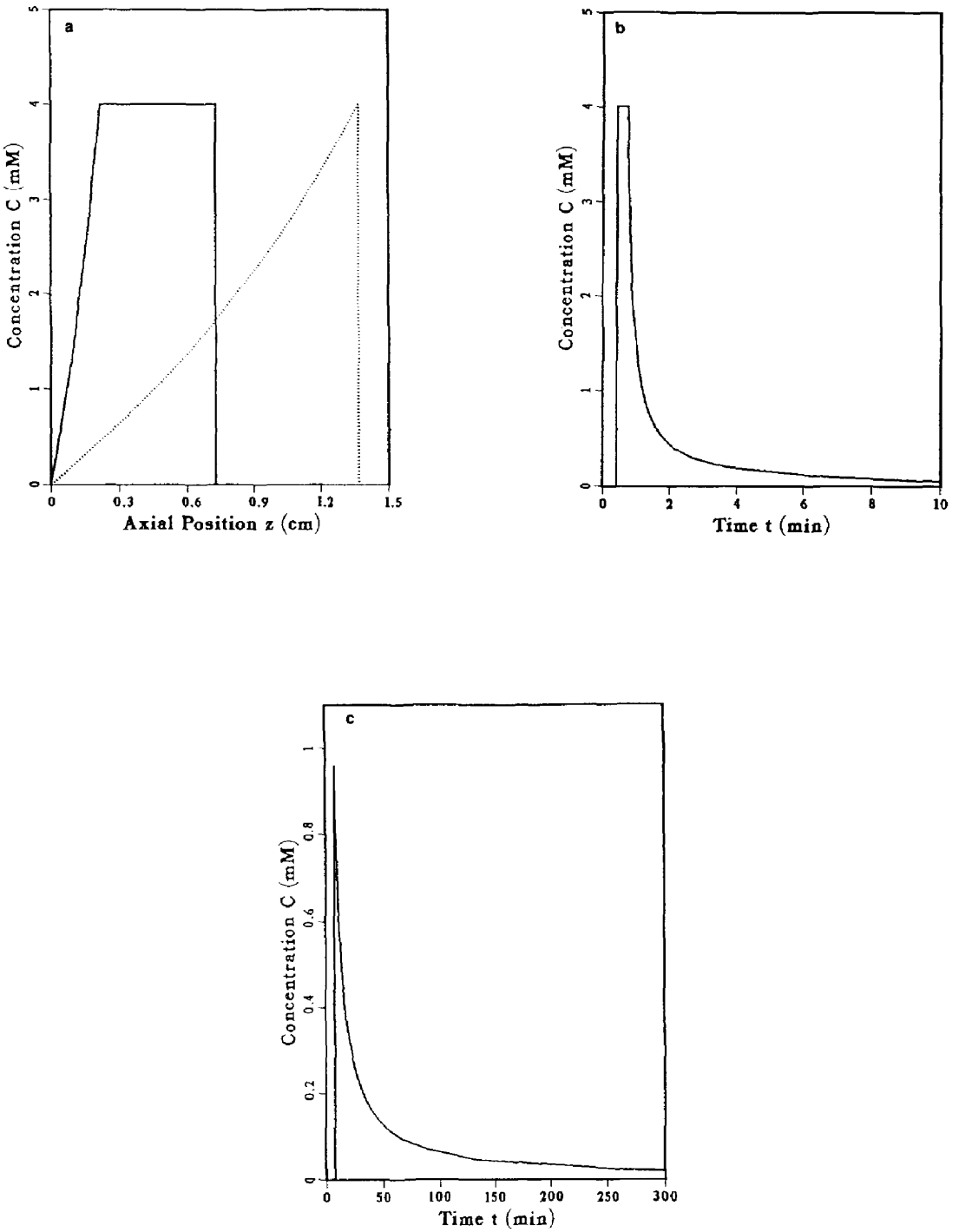

Fig. 4. Non-negative velocity case: $\beta = 4$, $C_0 = 4$ mM. Liquid-phase flow velocity, $u = 5$ cm/s; short column, $L = 0.5$ cm; long column, $L = 5$ cm; middle column, $L = 1.5$ cm, which is just shorter than the characteristic length L^*; two characteristic concentrations $C_1 = 5$ mM and $C_2 = 12.5$ mM. (a) Axial band profiles along the column at two different times: solid line, $t = 0.6$ s; dotted line, $t = 1.13$ s. (b) Temporal band profiles at the raffinate outlet ($z = L$), short column. (c) Same as (b) but long column.

profiles go through the origin. Accordingly, the end of the band, the concentration $C = 0$ will never exit from the column. The profile will tail indefinitely, an exceptional situation with the ideal model and its hyperbolic mass balance equation. The elution profile in the liquid phase is illustrated in Fig. 4b. All the other features of the band profiles are given by the same equations as in the previous case.

Third case: the limit velocity is negative, $\beta k'_0 > 1$

In this case, the retention of the component injected and/or the solid-phase velocity are high. Accordingly, the velocities associated with low concentrations are negative. However, because virtually all solid phases have a saturation capacity, equilibrium isotherms have a horizontal asymptote and dq/dC tends toward zero when C increases indefinitely. Hence there is always, at least in principle, a value of the injection concentration, C_0, which is high enough for the velocity associated with the high concentrations of the diffuse boundary and/or for the front shock of the rectangular injection pulse to have a positive velocity. Accordingly, we must distinguish several cases, depending on the signs of these velocities (see Fig. 5). Let C_1 be the concentration for which $u_z(C_1) = 0$ and C_2 the concentration for which $U_s(C_2) = 0$. C_1 is smaller than C_2 since the isotherm is convex upward and $U_s(C) < u_z(C)$ for all values of C, as illustrated in Fig. 5.

From Eqs. 15 and 16, we derive the equations giving C_1 and C_2:

$$\frac{dq}{dC}(C_1) = \frac{1}{\beta F} \tag{31a}$$

$$\frac{\Delta q}{\Delta C}(C_2) = \frac{1}{\beta F} \tag{31b}$$

For the Langmuir isotherm, these equations can be solved in closed form:

$$C_1 = \frac{1}{b}(\sqrt{\beta k'_0} - 1) \tag{31c}$$

$$C_2 = \frac{1}{b}(\beta k'_0 - 1) \tag{31d}$$

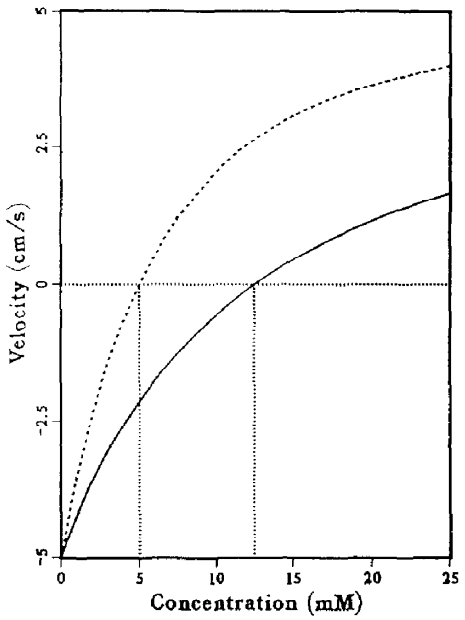

Fig. 5. Definition of the concentrations C_1 and C_2. Solid line, shock velocity, $U_s(C)$; dashed line, velocity associated with a concentration, $u_z(C)$. Experimental conditions as in Fig. 4.

and $C_1 < C_2$. In the figures, $\beta = 9$, $k'_0 = 0.25$, $b = 0.1$ mM^{-1}, so $C_1 = 5$ mM and $C_2 = 12.5$ mM. Note that $\beta = 9$ would probably be an unrealistically high value in many applications but this choice permits informative illustrations of the profiles obtained.

As illustrated in Fig. 5, there are three different cases:

(i) $C_0 < C_1$. The velocities associated with any possible concentrations of the band profiles are negative, $u_z(C) < 0$, $U_s(C) < 0$, $\forall C$. The band migrates in the direction of the solid phase and the component is entirely eluted as an extract [17].

(ii) $C_1 < C_0 < C_2$. The front shock of the rectangular injection pulse has a negative velocity, i.e., moves in the direction of the solid phase. However, the velocities associated with the high concentrations of the band profile ($C_1 < C < C_0$) during injection and immediately after are positive and these concentrations tend to move in the direction of the liquid phase. Concentrations lower than C_1 have a negative associ-

ated velocity and move in the direction of the solid phase.

(iii) $C_2 < C_0$. The concentration of the rectangular pulse injection is high, the front shock of the injection band moves in the direction of the liquid phase and the shock keeps moving in this direction after the front shock has been captured by the rear, diffuse boundary, as long as $C_2 < C_M$. On a diffuse boundary, the high concentrations $C > C_1$ move in the direction of the liquid phase and the low concentrations in the direction of the solid phase.

The first case is simple, the whole band moves in the direction of the solid phase and exits with it through the column entrance, as an extract. The other two cases are more complex. The band will begin moving in the direction of the liquid phase and part of it may reach the exit and elute as a raffinate if the column is short enough. However, because chromatography involves dilution, the band will eventually turn around if the column is long, start moving in the direction of the solid phase, and elute through the column entrance, as an extract.

$C_0 < C_1$, *all velocities are negative*. As all velocities are negative, all concentrations move backwards, in the direction of the solid phase, towards the extract exit. However, as the velocity associated with a concentration increases with increasing concentration, the lower concentrations have the larger backward velocity in absolute value. As a consequence, the low concentrations move backward faster than the high concentrations and the front shock of the rectangular injection pulse is unstable. It collapses into a front diffuse boundary. In contrast and for the same reason, the rear shock of the injection pulse is stable and it propagates as a rear shock.

Accordingly, the retention time of a concentration on the front diffuse boundary is given by the following equation:

$$t_R^\beta(C) = \frac{-L}{u_z} = t_0\left(\frac{F\cdot\frac{dq}{dC}+1}{\beta F\cdot\frac{dq}{dC}-1}\right) \overset{\text{Langmuir}}{=} -t_0\left[1+\frac{(1+\beta)k'_0}{(1+bC)^2-\beta k'_0}\right] \quad (32)$$

The equation differs slightly from Eq. 19, by the lack of the term t_p. This comes from the origin of the diffuse boundary, the collapse of the front shock of the injection. The concentration $C = 0$ exits first, with a retention time

$$t_R^\beta(0) = t_0\cdot\frac{1+k'_0}{\beta k'_0-1} \quad (32b)$$

If part of the injection plateau remains at the time of elution, the end of the band is eluted at time

$$t_S^\beta(C_0) = t_p + \frac{-L}{U_S} = t_p + t_0\left(\frac{F\cdot\frac{\Delta q}{\Delta C}+1}{\beta F\cdot\frac{\Delta q}{\Delta C}-1}\right) \overset{\text{Langmuir}}{=} t_p - t_0\left[1+\frac{(1+\beta)k'_0}{(1+bC_0)-\beta k'_0}\right] \quad (33a)$$

If the rear shock has been captured by the front diffuse boundary, the maximum concentration of the band, C_M, is given by the same mass balance equation (Eq. 21). In the case of a Langmuir isotherm (Eq. 22), the retention time of the shock can be solved in closed form and becomes

$$t_R^\beta(C_M) = -t_0 - t_0(1+\beta)k'_0 \times \frac{(1-L_f^\beta)^2}{L_f^\beta(1+\beta k'_0)+1-\beta k'_0+2\sqrt{L_f^\beta(1+L_f^\beta\beta k'_0-\beta k'_0)}} \quad (33b)$$

Note that in this case, the loading factor, L_f^β, has a negative value since the column length is negative in this direction.

In fact, the solution of the problem in this case is the exact opposite of the solution obtained in the case of a positive velocity, studied in the previous section, the roles of the front and rear shocks of the rectangular injection pulse being exchanged. Accordingly, the parameter t_p, the width of the rectangular injection pulse, appears in those equations of the present solution (negative velocities) whereas it does not in the equations of the other solution (positive velocities), and conversely.

Fig. 6a illustrates the concentration profiles of the band along the column at two different times after the injection. Fig. 6b shows the elution

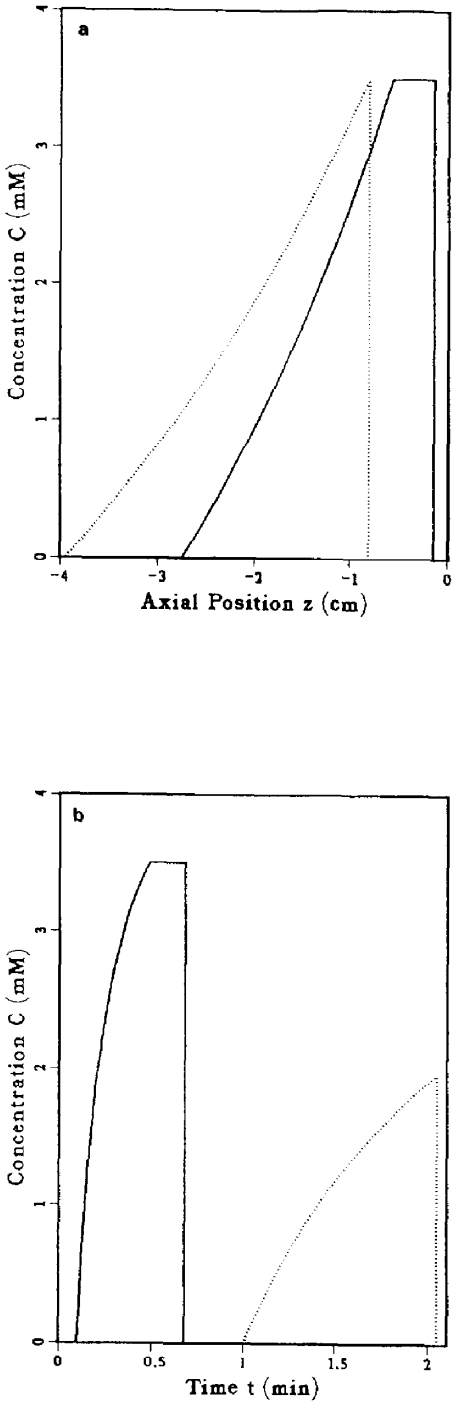

Fig. 6. Negative velocity case: $C_0 < C_1$, $\beta = 9$, $C_0 = 3.5$ mM. Experimental conditions as in Fig. 4. (a) Axial band profiles the column at two different times: solid line, $t = 0.55$ s; dotted line, $t = 0.67$ s. (b) Temporal band profiles at the extract outlet ($z = -L$): solid line, short column; dotted line, long column.

band profiles at the extract exit of a short (solid line) and a long column (dotted line). The former band profile exhibits a residual plateau at the injection concentration and the retention time of the rear shock is given by Eq. 33a. The latter profile is shorter and its maximum concentration is given by Eq. 21 or 33b.

$C_0 = C_1$, the velocity associated with the concentration C_0 is zero. This is a transition case. When the injection concentration C_0 is equal to C_1, $u_z(C_0) = u_z(C_1) = 0$. Hence the maximum concentration of a diffuse boundary does not move. The front shock of the injection pulse is unstable and a front diffuse boundary is formed as in the previous case. The retention time of a concentration C on this boundary is given by Eq. 32. The top point of this boundary, however, has a velocity equal to zero and it stays at the origin. Fig. 7a shows the concentration profile along the column at three successive times, corresponding to the injection still in progress, the end of the injection and some time afterward. The rear shock cannot move forward as its velocity is negative $[U_s(C) < u_z(C)]$. It cannot move backwards, however, because it is kept at the origin by the injection as long as it is in progress. In this case, there is no injection plateau. However long the injection or short the column, this plateau is reduced to a single point.

When the injection is finished, the rear of the band is a stable shock which has a negative velocity. The front diffuse boundary of the band and its rear shock have the same maximum concentration, C_0. At this concentration, $U_s(C_0) < u_z(C_0)$ and the velocity of the rear shock is lower than the velocity associated with the band maximum. As these velocities are negative, the shock starts moving backwards as soon as the feed injection ends. It moves backwards faster than the top of the diffuse boundary. Hence the tip of the shock (and the top of the band) is eroded progressively and the height of the concentration point at $z = 0$ begins to decrease. Since C_M becomes lower than $C_0 = C_1$, the velocity associated with C_M becomes negative and the band rear shock begins to move backward (Fig. 7a, third profile). The elution profiles are illustrated in Fig. 7b for two different

column lengths. These profiles are characterized by the lack of a plateau at the injection concentration, whatever the width or duration of the injection. Since the injection plateau has been eroded away, the maximum concentration of the band, C_M, is given by Eq. 21 in the general case or Eq. 22 for a Langmuir isotherm.

$C_1 \leq C_0 \leq C_2$, *the velocity of the shock is negative, the velocity associated with* C_0 *is positive.* When the injection concentration, C_0, is between C_1 and C_2, the velocity associated with the high concentrations on the diffuse boundary, $u_z(C)$, is positive whereas the velocity associated with the low concentrations is negative. For the intermediate concentration C_1, $u_z(C_1) = 0$. However, the shock velocity, $U_s(C)$ is always negative. This results in a more complex situation than previously encountered.

The rectangular pulse injected into the column begins with a front shock rising from 0 to C_0. This shock cannot move. If it moves backwards, it is unstable and collapses into a diffuse boundary whose highest concentration, C_0, has a positive velocity, and hence should move forward. However, if the concentration C_0 moves forwards it becomes the head of a stable front shock, a shock whose velocity is negative, so this shock should move backwards. Thus, it is physically impossible for the shock to move either forwards or backwards. As in the previous case, the band front does not move but it stays at the injection point.

The situation is the same for all concentrations between C_0 and C_1. These concentrations cannot move in the column. Concentrations lower than C_1 move backwards, as part of a front diffuse boundary, as they do in the first case since their velocities are negative. Concentrations larger than C_1 should move forwards since their associated velocities are positive. However, if they move forwards, they must be part of a stable front shock whose velocity must be negative [since $U_s(C_0) < 0$]. The paradoxical situation is that although a feed at concentration C_0 is pumped into the column, only concentrations lower than C_1 (or larger than C_2, see next

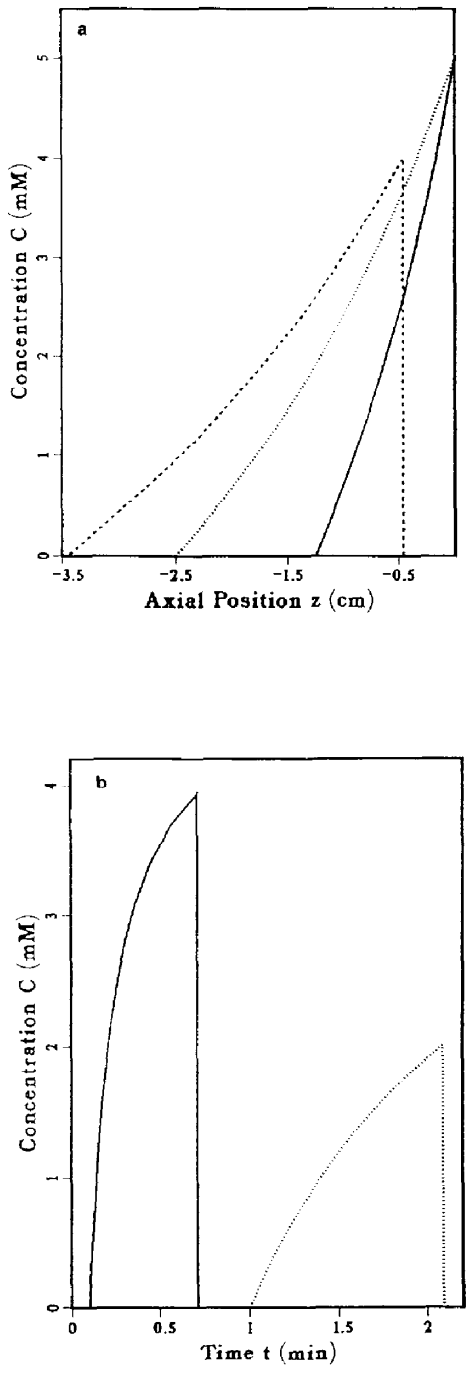

Fig. 7. Negative velocity case: $C_0 = C_1$, $\beta = 9$, $C_0 = 5.0$ mM. Experimental conditions as in Fig. 4. (a) Axial band profiles along the column at three different times: solid line, $t = 0.25$ s; dotted line, $t = 0.5$ s; dashed line, $t = 0.9$ s. (b) Temporal band profiles at the extract outlet ($z = -L$): solid line, short column; dotted line, long column.

section) can move along the column. The column offers an infinite resistance to its penetration by a feed at a concentration intermediate between C_1 and C_2. In other words, there is an instantaneous dilution of the feed from C_0 to C_1, certainly a phenomenon to be avoided.

When the injection is finished, the maximum concentration of the band begins to decrease, the amplitude of the rear shock becomes smaller than C_1 and it begins to move backwards at a velocity which increases progressively as the shock amplitude decreases with progressive dilution. The band profile is given by the same equations as in the previous case.

$C_0 > C_2$, the velocity of the injection shock is positive. When the injection concentration C_0 is higher than C_2, we can distinguish three concentration ranges in the boundaries of the band profile:

(i) when $C \leq C_1$, both the velocity associated with the concentration C, $u_z(C)$, and the shock velocity, $U_s(C)$, are negative;

(ii) when $C_1 \leq C \leq C_2$, the velocity associated with the concentration C is positive and the shock velocity is negative;

(iii) when $C_2 \leq C$, both the velocity associated with the concentration C and the shock velocity are positive.

Thus, the front shock of a rectangular injection pulse has a positive velocity, $U_s(C_0)$; it moves forwards and it is a stable shock. When the rectangular injection pulse ends, the concentration profile along the column is the same as in the case of a component for which the velocities associated with any values of the concentration are positive and whose band moves forward (*First case*). As in this case also, the rear shock is unstable and collapses, because the velocity associated with a concentration increases with increasing concentration. There is a major difference, however: the velocities associated with the low concentrations are negative and these concentrations move backwards while the front shock continues to move forwards. During the whole time when the injection is in process, the concentration C_1 remains stable at the origin. It cannot move as its velocity is zero (Fig. 8a).

As its front moves forwards and its rear moves backwards, the band spreads rapidly and the diffuse boundary captures the shock. The amplitude of the shock begins to decrease and it slows. In contrast to the case when $\beta k'_0 < 1$ (*First case*), the shock velocity can become zero in the present case. This takes place when the shock amplitude becomes equal to C_2. Then the shock stops and starts moving backwards. $u_z(C)$ is still positive, however, and the amplitude of the shock decreases rapidly. As long as the shock amplitude is larger than C_1, the diffuse boundary has the concentration C_1 at the injection point ($z = 0$). The rear shock goes by the injection point when its amplitude becomes equal to C_1. Spatial band profiles observed shortly after the end of the injection and at the time when the diffuse boundary captures the shock are illustrated in Fig. 8a and b.

Depending on the duration of the injection and the column length, part of the feed may exit with the liquid phase, as a raffinate, if the shock can reach $z = L$ during its forward migration, i.e., if the amplitude of the shock when it reaches the exit of the column is such that $C > C_2$. The rest of the injection leaves the column with the solid phase, as an extract. Thus, three types of band profiles can be obtained:

(a) If the column is short and/or the injection duration long enough, the front shock at C_0 reaches the column raffinate exit ($z = L$), followed by the rest of the injection plateau and a diffuse boundary. As the injection plateau is not yet eroded, the retention time of the shock is given by

$$t_s^{\beta}(C_0) = t_0\left[\frac{1 + F \cdot \frac{\Delta q}{\Delta C}(C_0)}{1 - \beta F \cdot \frac{\Delta q}{\Delta C}(C_0)}\right] \overset{\text{Langmuir}}{=} t_0\left[1 + \frac{(1+\beta)k'_0}{(1+bC_0) - \beta k'_0}\right] \quad (34)$$

The diffuse boundary of the elution profile is derived from the retention time of a concentration

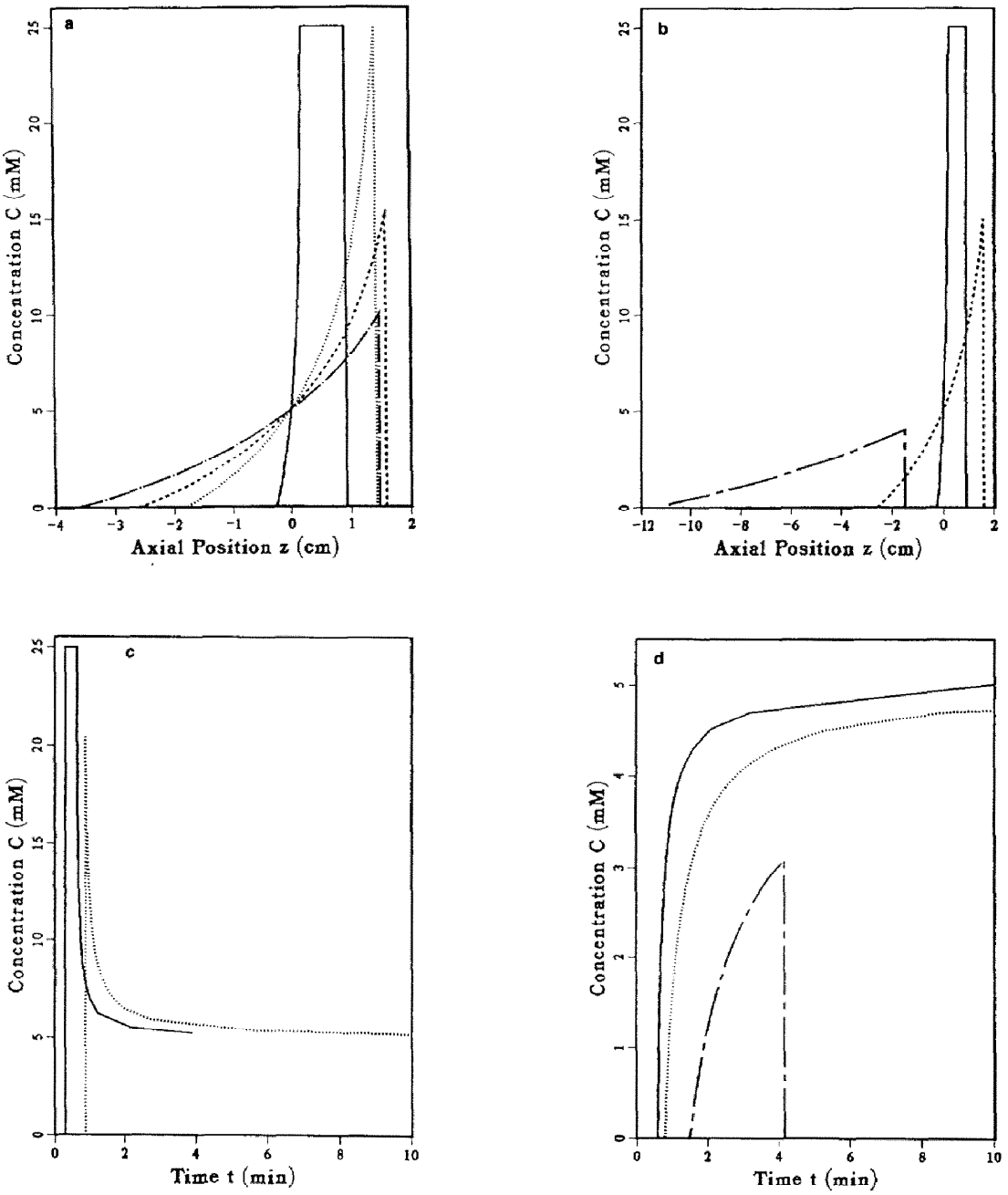

Fig. 8. Negative velocity case: $C_0 > C_2$, $\beta = 9$, $C_0 = 25.0$ mM. Experimental conditions as in Fig. 4. (a) Axial band profiles along the column at four different times: solid line, $t = 0.55$ s; dotted line, $t = 0.86$ s; dashed line, $t = 1.02$ s; chain-dotted line $t = 1.22$ s. $L^* = 1.61$ cm. (b) Axial band profiles along the column at three different times: solid line, $t = 0.55$ s; dashed line, $t = 1.02$ s; chain-dashed line, $t = 1.61$ s. (c) Temporal band profiles consisting of the front shock and the higher part of the rear diffuse boundary at the raffinate outlet ($z = L$): solid line, short column; dotted line, middle column. (d) Temporal band profiles at the extract outlet ($z = -L$): solid line, short column; dotted line, middle column. For both columns, the profiles consist of only the lower part of the rear diffuse boundary. Chain-dashed line, long column; the profile consists of the whole band (the front shock and the whole rear diffuse boundary).

$$t_R^{\beta}(C) = t_p + t_0\left(\frac{F\cdot\frac{dq}{dC}+1}{1-\beta F\cdot\frac{dq}{dC}}\right)$$

$$\overset{\text{Langmuir}}{=} t_p + t_0\left[1+\frac{(1+\beta)k_0'}{(1+bC)^2-\beta k_0'}\right]$$

$$(C \geqslant C_1) \qquad (35)$$

This equation is the same as Eq. 19. The rear of the injection plateau is eluted at time $t_R^{\beta}(C_0)$. If this time is shorter than the retention time of the shock, the latter has been captured by the diffuse boundary and the amplitude of the shock is less than C_0 (see case b).

Only concentrations larger than C_1 can leave the column through the liquid-phase exit (raffinate). Lower concentrations are associated with negative velocities and must leave the column through the other end (extract). Fig. 8c and d show the elution profiles at the raffinate (Fig. 8c) and extract (Fig. 8d) exits. In Fig. 8c, two possibilities are shown, when the profile includes a plateau at the injection concentration and when this plateau has disappeared. These elution profiles never end but have a horizontal asymptote at $C = C_1$ since the retention time of concentration C_1 is infinite. Although the liquid-phase concentration profile has an infinite area, there is no physical impossibility because the outgoing mass flux of the component falls rapidly towards zero when C tends towards C_1 and is zero at $C = C_1$. The area of the component-concentration profile in the liquid phase has no direct physical meaning and is not conservative because there is a finite concentration, $q_1 = f(C_1)$, in the solid phase entering the column at $z = L$ [1]. Only the net mass flux of component leaving the column is conserved. Note the striking difference between the profiles in Fig. 8d, depending on whether the column is or is not long enough for the front shock to change its direction of migration. When the whole sample exits as an extract (chain-dashed line) the elution profile appears normal. When the column is short and the front shock elutes at the raffinate exit, the elution profile has an infinite area. This is a good illustration of the fact that only the mass flux is conservative, while the area of the concentration profile is meaningless in moving-bed chromatography.

The concentrations which are lower than C_1 leave the column with the solid phase at the extract exit ($z = -L$). Typical profiles are shown in Fig. 8d. The equation of the continuous profile is not Eq. 35, but

$$t_R^{\beta}(C) = t_p + \frac{-L}{u_z} = t_p - t_0\left(\frac{1+F\cdot\frac{dq}{dC}}{1-\beta F\cdot\frac{dq}{dC}}\right)$$

$$\overset{\text{Langmuir}}{=} t_p - t_0\left[1+\frac{(1+\beta)k_0'}{(1+bC)^2-\beta k_0'}\right]$$

$$(C \leqslant C_1) \qquad (36)$$

Although similar to Eqs. 32 and 35, this equation differs from Eq. 32 by the presence of t_p and from Eq. 35 by the sign of L. The concentration C_1 remains stagnant anywhere it appears on a diffuse boundary, first at the injection point, then all along the column. It can be eluted only if it is swept out by the passage a shock. If, as in the present case, no shock comes to purge the column, it becomes filled with the component at concentration C_1, a concentration which does not move in either direction.

(b) As we have shown earlier, the amplitude of the shock decreases once it has been captured by the diffuse boundary and the shock slows. Eventually, its velocity becomes zero when the band height has decreased to C_2. Then, the shock turns around to migrate in the opposite direction, pushing the entire band profile backward. Thus, there is a critical column length, L^*, such that if $L < L^*$, the front shock exits with the liquid phase, and some raffinate is collected, whereas if $L > L^*$, the same shock exits, as the rear band shock, through the column entrance, with the solid phase and all the sample is collected as an extract. This critical column length is derived from the integral mass balance equation (Eq. 21), by writing that the maximum concentration of the band, or shock amplitude, C_M, has become equal to C_2. This length is

$$L^* = \frac{t_p u[C_0 - \beta F q(C_0)]}{C_2 + F q(C_2)} \overset{\text{Langmuir}}{=} \frac{t_p u C_0 \beta}{C_2(1+\beta)}\left(1 - \frac{\beta k'_0}{1 + bC_0}\right) \quad (37)$$

If the column length is less than L^*, the maximum band concentration C_M is larger than C_2 and the situation is similar (see Fig. 7b, dotted line) to that just described in (a). The maximum band C_M can be obtained from the mass balance, i.e., the injected mass at the feed point must be equal to the sum of the masses leaving the column at both the positive and negative outlets and the mass staying inside the column. Hence C_M is determined by Eq. 21 or 22, and the retention times are given by Eq. 35.

(c) If the column is longer than L^*, the band maximum C_M decreases and becomes lower than C_2 before it can leave the column at its raffinate outlet. In this case, $U_s(C)$ changes sign at concentration C_2 and the front shock stops and changes the direction of its migration from forwards to backwards. This shock remains stable after shifting the direction of its migration. It now moves backwards, in the direction of the solid phase, opposite to the direction of migration of the high concentrations ($C_1 < C < C_2$) on the diffuse boundary. This interaction between the shock and the high concentrations on the diffuse boundary causes a rapid decay of the shock. During all the time when the band shock is in the positive part of the column (i.e., $z > 0$), the diffuse boundary of the band has a concentration equal to C_1 at the origin ($z = 0$). Hence the shock concentration is equal to C_1 when it passes the feed point. Further, concentrations on the diffuse boundary can move backwards only if $C < C_1$. Eventually, the front shock leaves the column at the extract outlet. In this last case, the whole band exits from the column at the extract end and is not split into two fractions as it is in the cases (a) and (b) above.

As the whole band leaves the column at the negative (extract) outlet, $z = -L$, the retention times of the front diffuse boundary is given by Eq. 36, and the band maximum C_M is given by

$$-t_p[C - \beta F q(C)]\Big|_0^{C_0} = t_0\left\{[C + F q(C)]\Big|_0^{C_M} \frac{1 + F \cdot \frac{dq}{dC}(C_M)}{1 - \beta F \cdot \frac{dq}{dC}(C_M)}[C - \beta F q(C)]\Big|_0^{C_M}\right\} \quad (38)$$

For the Langmuir isotherm, C_M can be expressed analytically:

$$C_M = \frac{\sqrt{L_f^\beta[-1 + (L_f^\beta + 1)\beta k'_0]} - L_f^\beta}{b(1 + L_f^\beta)} \quad (39)$$

and the retention time of the band is

$$t_R^\beta(C_M) = t_p - t_0 - t_0(1+\beta)k'_0 \times \frac{(1 + L_f^\beta)^2}{-L_f^{0\beta}(1 + \beta k'_0) + 1 - \beta k'_0 + 2\sqrt{L_f^\beta(-1 + L_f^\beta \beta k'_0 + \beta k'_0)}} \quad (40)$$

Note that, in this case, the rear of the injection profile reaches the column end first, before the front shock of the injection. This is due to the change in the propagation direction of the front shock which results in a negative sign in Eq. 38 replacing a positive sign in Eq. 21. Whereas in all the other cases the column operates as a first in, first out storage device, in this case it operates on the first in, last out principle.

Symbols

a	numerical coefficient in the Langmuir isotherm
b	numerical coefficient in the Langmuir isotherm
C	liquid concentration of the component
C_M	maximum concentration of the profile
C_0	component concentration in the feed
C_1	characteristic concentration [$u_z(C_1) = 0$]
C_2	characteristic concentration [$U_s(C_2) = 0$]
F	phase ratio [$F = (1 - \varepsilon)/\varepsilon$]
F_v	liquid-phase flow-rate ($F_v = \varepsilon S u$)

g^0	net mass flux
$k'_0 = \mathrm{Fa}$	limit retention factor at infinite dilution
L	half the column length
L_f	loading factor or ratio of the sample size to the column saturation capacity
L_s	length of each sub-column or column sub-section in SMB
L^*	critical column length (Eq. 37)
M^{in}	mass flux entering the column
M^{out}	mass flux leaving the column
n	amount injected ($n = C_0 t_p F_v$)
q	solid-phase concentration of the component
S	column cross-sectional area or area of the band profile
t	time
t_p	injection duration
$t_R(C_M)$	retention time of the shock
t_s	switching interval time
U_s	shock velocity
u	flow velocity of the liquid phase
u^{SMB}	velocity of the fluid phase in SMB
$u_z(C)$	velocity associated with a concentration C on a diffuse boundary
V^β	migration velocity of an infinitely thin slice of the profile of a retained component
v	velocity of the moving solid phase
v^{SMB}	velocity of the solid phase in SMB
z	position in the column

Greek letters

β	ratio of the velocities of the solid and the liquid phase in the moving bed column ($\beta = v/u$)
γ	intermediate parameter, $\gamma = \sqrt{\beta k'_0}$
ΔC	concentration amplitude of the shock in the liquid phase
Δq	concentration amplitude of the shock in the solid phase
ε	total column porosity

Superscript

β	characterizes parameters used in countercurrent chromatography

Acknowledgements

This work was supported in part by Grant CHE-92-01663 of the National Science Foundation and by the cooperative agreement between the University of Tennessee and the Oak Ridge National Laboratory. We acknowledge the continuous support of our computational effort by the University of Tennessee Computing Center.

References

[1] G. Ganetsos and P.E. Baker, *Preparative and Production Scale Chromatography*, Marcel Dekker, New York, 1993.
[2] D.M. Ruthven, *Chem. Eng. Sci.* 44 (1989) 1011.
[3] B.B. Fish, R.W. Carr and R. Aris, *AIChE J.* 35 (1989) 80.
[4] B. Balannec and G. Hotier, *Rev. Inst. Fr. Pet.*, 46 (1991) 803.
[5] P.C. Wankat, *Large-Scale Adsorption and Chromatography*, CRC Press, Boca Raton, FL, 1986.
[6] G. Storti, M. Masi, S. Carra and M. Morbidelli, *J. Prep. Chromatogr.*, 1 (1988) 1.
[7] G. Guiochon, S. Golshan-Shirazi and A.M. Katti, *Fundamentals of Preparative and Nonlinear Chromatography*, Academic Press, Boston, MA, 1994.
[8] E. Glueckauf, *Proc. R. Soc. London, Ser. A.*, 186 (1946) 35.
[9] F. Helfferich and G. Klein, *Multicomponent Chromatography—A Theory of Interference*, Marcel Dekker, New York, 1970.
[10] R. Aris and N.R. Amundson, *Mathematical Methods in Chemical Engineering*, Prentice-Hall, Englewood Cliffs, NJ, 1973, Vol. 2.
[11] H.-K. Rhee, R. Aris and N.R. Amundson, *First-Order Partial Differential Equations. II. Theory and Applications of Hyperbolic Systems of Quasilinear Hyperbolic Equations*, Prentice-Hall, Englewood Cliffs, NJ, 1986.
[12] S. Golshan-Shirazi and G. Guiochon, *Anal. Chem.*, 60 (1988) 2364.
[13] B. Lin, S. Golshan-Shirazi, Z. Ma and G. Guiochon, *Anal. Chem.*, 60 (1988) 2647.
[14] G. Guiochon, S. Ghodbane, S. Golshan-Shirazi, J.-X. Huang, A.M. Kati, B. Lin and Z. Ma, *Talanta*, 36 (1989) 19.
[15] B. Lin, Z. Ma, S. Golshan-Shirazi and G. Guiochon, *J. Chromatogr.*, 500 (1990) 185.
[16] S. Golshan-Shirazi and G. Guiochon, *J. Phys. Chem.*, 94 (1990) 495.
[17] D.B. Broughton, *Sep. Sci.*, 19 (1984) 723.